Last Time: Limits

Recall Curves Criterion: - A function f of several variables satisfies $\lim_{\vec{x}\to\vec{a}} f(\vec{x}) = L$ if and only if for all continuous space curves $\vec{r}(t)$ with $\lim_{t\to\infty} \vec{r}(t) = \vec{a}$ and $\vec{r}(t) \neq \vec{a}$ for all t we have $\lim_{t\to\infty} f(\vec{r}(t)) = L$.

- We saw an example showing $\lim_{\vec{x}\to\vec{a}} f(\vec{x})$ DNE...

Idea: Find two curves $\vec{r}_0(t)$ and $\vec{r}_1(t)$ with $\lim_{t\to\infty} \vec{r}_i(t)$ and $\lim_{t\to\infty} f(\vec{r}_0(t)) \neq \lim_{t\to\infty} f(\vec{r}_1(t))$

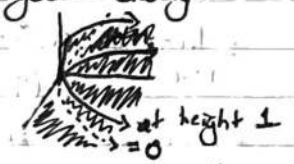
NB: The collection of lines

$$l_{a,b}(t) = \vec{a} + t\langle a, b\rangle$$

is a great collection of curves to work with to solve b/c $\lim_{t\to 0} l_{a,b}(t) = \vec{a}$

NB: These are not always good enough

Ex: Let $f(x,y) = \begin{cases} 1 & \text{if } y = x^2 \\ 0 & \text{otherwise} \end{cases}$



Claim: $\lim_{\vec{x}\to\vec{0}} f(\vec{x})$ DNE

- For lines $l_{a,b}(t)$:

$$\lim_{t\to 0^+} f(l_{a,b}(t)) = \lim_{t\to 0^+} f(at, bt)$$

off the origin, (at, bt) satisfies $bt = (at)^2$ at

IF: $f(at, bt) = 0$ for all but $\leq ab$ many times

IF: $\lim_{t\to 0^+} f(l_{a,b}(t)) = 0$

- On the other hand, letting $\vec{r}(t) = \langle t, t^2\rangle$, we see

$f(\vec{r}(t)) = f(t, t^2) = 1$ for all t.

IF: $\lim_{t\to 0} f(\vec{r}(t)) = \lim_{t\to 0} 1 = 1$

- Thus since $0 \neq 1$, $\lim_{\vec{x}\to\vec{0}} f(\vec{x})$ DNE!

Q: How can we show when limits do exist?

A Trick: Try polar coordinates...

Ex: Does $\lim_{(x,y)\to(0,0)} \frac{\sin(x^2+y^2)}{x^2+y^2}$ exist?

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Sol: Lets convert to polar coords! $\begin{cases} x = r\cos\theta \\ y = r\sin\theta \\ (x,y)\to(0,0) \Rightarrow r\to 0^+ \end{cases}$

$$\lim_{(x,y)\to(0,0)} \frac{\sin(x^2+y^2)}{x^2+y^2} \overset{\text{if it exists}}{=} \lim_{r\to 0^+} \frac{\sin((r\cos\theta)^2+(r\sin\theta)^2)}{(r\cos\theta)^2+(r\sin\theta)^2}$$

$$= \lim_{r\to 0^+} \frac{\sin(r^2(\cos^2\theta+\sin^2\theta))}{r^2(\cos^2\theta+\sin^2\theta)}$$

$$= \lim_{r\to 0^+} \frac{\sin(r^2)}{r^2} \to \frac{0}{0}\text{-type}$$

$$\boxed{LH} = \lim_{r\to 0^+} \frac{2r\cos(r^2)}{2r} = \lim_{r\to 0^+} \cos(r^2) = \cos(0^2) = 1$$

Ex: Does $\lim_{(x,y)\to(0,0)} \frac{x^2-y^2}{x^2+y^2}$ exist?

Sol: $\lim_{(x,y)\to(0,0)} \frac{x^2-y^2}{x^2+y^2} = \lim_{r\to 0^+} \frac{(r\cos\theta)^2-(r\sin\theta)^2}{(r\cos\theta)^2+(r\sin\theta)^2}$

$$= \lim_{r\to 0^+} \frac{r^2(\cos^2\theta-\sin^2\theta)}{r^2}$$

$$= \lim_{r\to 0^+} \cos^2\theta - \sin^2\theta$$

$$= \lim_{r\to 0^+} \cos(2\theta) = \cos(2\theta)$$

(Dependent on θ)

If we approach along an angle of $\theta = \pi/2$, we expect $\lim_{\vec{x}\to\vec{0}} f(\vec{x}) = \cos(2\cdot\pi/2) = -1$ Where as approaching at $\theta = 0$ yields $\lim_{\vec{x}\to\vec{0}} f(\vec{x}) = \cos(2\cdot 0) = 1$

If: $\lim_{r\to 0^+} \frac{x^2-y^2}{x^2+y^2}$ DNE!

Def: A function f of n-variables is continuous at $\vec{a} \in \text{dom}(f)$ when $\lim_{\vec{x}\to\vec{a}} f(\vec{x}) = f(\vec{a})$

f is continuous on set D when f is cts at every value of D

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Ex: Every polynomial in n-variables is cts on $\mathbb{R}^n$

Ex: Every rational function of n-variables is cts on its domain

E.g. $\frac{x^2-y^2}{x^2+y^2}$ is cts on its domain

i.e. it is cts everywhere but $(0,0)$.

Ex: $\frac{\sin(x^2+y^2)}{x^2+y^2}$ is cts everywhere but $(0,0)$.

OTOH: $f(x,y) = \begin{cases} \frac{\sin(x^2+y^2)}{x^2+y^2} & \text{if } (x,y) \neq (0,0) \\ 1 & \text{if } (x,y) = (0,0) \end{cases}$ is cts everywhere (why?)

NB: Usual "rules" for continuity apply. (Calc I)

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Derivatives of Multivariable Functions

Idea: The derivative measures change in output from corresponding small changes in input...
IN SOME DIRECTION

Def: Let f be a function of n-variables and $\vec{u}$ a unit vector in $\mathbb{R}^n$. Let $\vec{a} \in \text{dom}(f)$.
- The directional derivative of f at $\vec{a}$ in direction of $\vec{u}$ is $D_{\vec{u}} f(\vec{a}) = \lim_{h \to 0^+} \frac{f(\vec{a}+h\vec{u}) - f(\vec{a})}{h}$

Ex: Compute the directional derivative of $f(x,y) = xy$ at $\vec{a} = \langle 1, 3 \rangle$ in direction $\vec{u} = \frac{1}{2}\langle \sqrt{2}, \sqrt{2} \rangle$

Sol: $D_{\vec{u}} f(\vec{a}) = \lim_{h \to 0^+} \frac{f(\vec{a}+h\vec{u}) - f(\vec{a})}{h}$

$$= \lim_{h \to 0^+} \frac{f(1+\frac{\sqrt{2}}{2}h, 3+\frac{\sqrt{2}}{2}h) - f(1,3)}{h}$$

$$= \lim_{h \to 0^+} \frac{(1+\frac{\sqrt{2}}{2}h)(3+\frac{\sqrt{2}}{2}h) - 1\cdot 3}{h}$$

$$= \lim_{h \to 0^+} \frac{3 + h(\frac{3\sqrt{2}}{2} + \frac{\sqrt{2}}{2}) + h^2 - 3}{h}$$

$$= \lim_{h \to 0^+} \frac{h(2\sqrt{2}+h)}{h} = \lim_{h \to 0^+} (2\sqrt{2}+h) = 2\sqrt{2} + 0 = \boxed{2\sqrt{2}}$$

Exercise: Repeat example with $\vec{a} = \langle x, y \rangle$

NB: The directional derivative is very general.
We want something like the "rules" from Calc I...

Def: Let f be a function of n-variables and let $\vec{e}_k$ be "k-th standard basic vector in $\mathbb{R}^n$"
i.e. $\vec{e}_k = \langle 0, 0, \ldots, \underset{\uparrow\, k^{th} \text{ position}}{1}, \ldots, 0 \rangle$
- The "k^{th} partial derivative of f (alt. the partial derivative of f wrt x_k) is $D_{\vec{e}_k} f(\vec{a})$